MATHEMATICS

ZERO DIMENSIONAL BITOPOLOGICAL SPACES

BY

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1. *Introduction*

A bitopological space is a triple $(X, \mathscr{T}_1, \mathscr{T}_2)$ where $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on a set X. KELLY [3] initiated the systematic study of such spaces, and several other authors have contributed to the subsequent development of various bitopological properties. The purpose of this paper is to introduce the notion of bitopological zero dimensionality. Section 2 gives the basic definitions and results. In particular, it relates the concept of bitopological zero dimensionality to the ideas of total disconnectedness for bitopological spaces examined by SWART [10]. In section 3 we prove that the bitopological space induced by a non-archimedean quasi-metric is pairwise zero dimensional. This result enables us to relate this work to a recent quasi-metrization theorem for topological spaces obtained by FLETCHER and LINDGREN [2]. In particular, we try to reformulate as a bitopological problem their question as to whether their theorem provides necessary and sufficient conditions for quasi-metrizability of a topological space. Terms and notation not explained in this paper are taken from KELLY [3] and PERVIN [7].

2. *Bitopological zero dimensionality*

Definition. In the bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$, $\mathscr{T}_1$ is zero dimensional with respect to $\mathscr{T}_2$ if $\mathscr{T}_1$ has a base of $\mathscr{T}_2$ closed sets, that is, if for each point x in X and each $\mathscr{T}_1$ open set U containing x there is a $\mathscr{T}_2$ closed $\mathscr{T}_1$ open set G such that $x \in G \subset U$.

$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise zero dimensional if $\mathscr{T}_1$ is zero dimensional with respect to $\mathscr{T}_2$ and $\mathscr{T}_2$ is zero dimensional with respect to $\mathscr{T}_1$.

It follows immediately from the definition that if $\mathscr{T}_1$ is zero dimensional with respect to $\mathscr{T}_2$ then $\mathscr{T}_1$ is regular with respect to $\mathscr{T}_2$, see KELLY [3].

The following example shows that the bitopological zero dimensionality of $(X, \mathscr{T}_1, \mathscr{T}_2)$ is not related to the topological zero dimensionality of $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$.

Example 1. Let R be the set of real numbers, $\mathscr{L}$ be the left hand topology on R, and $\mathscr{R}$ be the right hand topology on R, see Pervin [7,

p. 50]. So $\mathscr{L}$ has base the family of sets $\{(-\infty, x): x \in R\}$ and $\mathscr{R}$ has the family $\{(x, +\infty): x \in R\}$ as base. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be the set Q of rationals regarded as a subspace of $(R, \mathscr{L}, \mathscr{R})$. Now if a is irrational, $Q \cap (-\infty, a] = Q \cap (-\infty, a)$, so that $\mathscr{T}_1$ has a base of $\mathscr{T}_2$ closed sets, namely $\{Q \cap (-\infty, a)\colon a \text{ is irrational}\}$. Similarly $\mathscr{T}_2$ is zero dimensional with respect to $\mathscr{T}_1$, so $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise zero dimensional. However, $(X, \mathscr{T}_1)$ is not a zero dimensional topological space, and nor is $(X, \mathscr{T}_2)$.

Murdeshwar and Naimpally [5] define $(X, \mathscr{T}_1, \mathscr{T}_2)$ to be pairwise T_0 (we will write MN pairwise T_0) if for each pair of distinct points of X there is a set which is either $\mathscr{T}_1$ open or $\mathscr{T}_2$ open containing one of the points but not the other. Fletcher, Hoyle and Patty [1] have a stronger definition. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is called pairwise T_0 (we will write FHP pairwise T_0) if for each pair x, y of distinct points in X there is either a $\mathscr{T}_1$ open set U such that $x \in U$ and $y \notin U$ or a $\mathscr{T}_2$ open set V such that $y \in V$ and $x \notin V$. Notice that the space $(X, \mathscr{T}_1, \mathscr{T}_2)$ described in Example 1 is MN pairwise T_0 but not FHP pairwise T_0.

We can now relate our notion of bitopological zero dimensionality to the ideas of total disconnectedness introduced by Swart [10, Definition 2.3].

Theorem 1. If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is FHP pairwise T_0 and pairwise zero dimensional then it is totally disconnected.

Proof. Let x and y be distinct points in X. Then there is
(i) a $\mathscr{T}_1$ open set U such that $x \in U$, $y \notin U$ or
(ii) a $\mathscr{T}_2$ open set V such that $x \notin V$, $y \in V$.
Since $\mathscr{T}_1$ is zero dimensional with respect to $\mathscr{T}_2$, in case (i) there is a $\mathscr{T}_1$ open $\mathscr{T}_2$ closed set G such that $x \in G \subset U$. Then $X = G/X - G$ is a separation of X with $x \in G$, $y \in X - G$, so that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is totally disconnected. Case (ii) follows similarly interchanging the roles of $\mathscr{T}_1$ and $\mathscr{T}_2$.

The next result is a weaker version of the previous one. Its proof is analogous to that above, except that it is longer in that there are four cases to discuss.

Theorem 2. If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is MN pairwise T_0 and pairwise zero dimensional then it is weakly totally disconnected.

Similar results using stronger separation properties but a weaker zero dimensional requirement, namely $\mathscr{T}_1$ zero dimensional with respect $\mathscr{T}_2$ in place of pairwise zero dimensional, have been obtained by Swart [10, Theorems 2.6 and 2.5].

We use the concept of bitopological local compactness discussed by the author in [8] to obtain a partial converse to the previous results, in Corollary 1 below.

Theorem 3. If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is totally disconnected and $\mathscr{T}_1$ is locally compact with respect to $\mathscr{T}_2$ then $\mathscr{T}_1$ is zero dimensional with respect to $\mathscr{T}_2$.

Proof. Let $x \in G \in \mathscr{T}_1$. Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is totally disconnected it is pairwise Hausdorff, so that by Proposition 2 of [8] there is a $\mathscr{T}_1$ open set V such that $x \in V \subset \mathscr{T}_2 \text{ cl } V \subset G$ and $\mathscr{T}_2 \text{ cl } V$ is pairwise compact. (Throughout this paper $\mathscr{T}$ cl A denotes the $\mathscr{T}$ closure of A.) If $V = \mathscr{T}_2 \text{ cl } V$, then V is $\mathscr{T}_2$ closed and there is nothing more to prove. Otherwise, let $W = \mathscr{T}_2 \text{ cl } V$ and consider the subspace $(W, \mathscr{T}_1{}^*, \mathscr{T}_2{}^*)$. Then $W - V$ is a non-empty proper $\mathscr{T}_1{}^*$ closed subset of the pairwise compact W, so that $W - V$ is $\mathscr{T}_2{}^*$ compact. For each point y in $W - V$ there is a separation $A_y|B_y$ of X with $x \in A_y$, $y \in B_y$ and $B_y \mathscr{T}_1$ closed and $\mathscr{T}_2$ open. Let $U_y = W \cap B_y$. Then the family $\{U_y : y \in W - V\}$ is a $\mathscr{T}_2{}^*$ open cover of $W - V$, so that it has a finite subcover, denoted by $U_1, \ldots, U_n$. If $U = \bigcup_{i=1}^n U_i$, then U is $\mathscr{T}_1{}^*$ closed, $\mathscr{T}_2{}^*$ open and $W - V \subset U$. Then we have $x \in W - U \subset V \subset W \subset G$, and $W - U$ is $\mathscr{T}_1$ open and $\mathscr{T}_2$ closed since V is $\mathscr{T}_1$ open and W is $\mathscr{T}_2$ closed. Hence $\mathscr{T}_1$ has a base of $\mathscr{T}_2$ closed sets as desired.

Corollary 1. If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is totally disconnected and pairwise locally compact then it is pairwise zero dimensional.

Theorem 2.7 of Swart [10] now follows as an immediate corollary.

3. *Non-archimedean quasi-metrics*

By a quasi-pseudo-metric on a set X we mean a non-negative real valued function p on XxX which vanishes on the diagonal and satisfies the triangle inequality in the form

$$p(x, y) \leqslant p(x, z) + p(z, y) \text{ for all } x, y, z \in X.$$

If p also satisfies $p(x, y) = 0$ implies $x = y$, then p is a quasi-metric. If p satisfies the stronger triangle inequality $p(x, y) \leqslant \max\{p(x, z), p(z, y)\}$ for all $x, y, z \in X$, then p is non-archimedean.

If p is a quasi-(pseudo)-metric on X, so is its conjugate q which is defined by $q(x, y) = p(y, x)$ for $x, y \in X$. Moreover, p is non-archimedean if and only if q is non-archimedean. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is quasi-(pseudo)-metrizable if there is a pair of conjugate quasi-(pseudo)-metrics p and q on X such that $\mathscr{T}_1 = \mathscr{T}(p)$ and $\mathscr{T}_2 = \mathscr{T}(q)$, where $\mathscr{T}(p)$ is the topology on X having the family $\{B(x, p, \varepsilon) : x \in X, \varepsilon > 0\}$ of p-balls as a base, where $B(x, p, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$.

It is well known that a non-archimedean metric space is zero dimensional, see Monna [4] for example. The bitopological situation is described by the following result.

Theorem 4. If a bitopological space is non-archimedeanly quasi-pseudo-metrizable then it is pairwise zero dimensional.

Proof. Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be induced by the pair of conjugate non-archimedean quasi-pseudo-metrics p and q on X. We show that any p-ball

$B(x, p, \varepsilon)$ is $\mathscr{T}_2$ closed. Let $y \in X - B(x, p, \varepsilon)$, so that $p(x, y) \geqslant \varepsilon$. Assume $c \in B(x, p, \varepsilon) \cap B(y, q, \varepsilon)$. Then $q(y, x) \leqslant \max\{q(y, c), q(c, x)\} < \varepsilon$, since $q(y, c) < \varepsilon$ and $q(c, x) = p(x, c) < \varepsilon$. But $q(y, x) = p(x, y) \geqslant \varepsilon$. Hence there is no such point c. Thus $B(x, p, \varepsilon) \cap B(y, q, \varepsilon) = \emptyset$.

For each point $y \in X - B(x, p, \varepsilon)$ we have $y \in B(y, q, \varepsilon) \subset X - B(x, p, \varepsilon)$, so that $X - B(x, p, \varepsilon)$ is $\mathscr{T}_2$ open. Similarly, $\mathscr{T}_2$ has a base of $\mathscr{T}_1$ closed sets, so that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise zero dimensional.

The problem of giving necessary and sufficient conditions for the quasi-metrizability of a topological space has been considered by several authors. NORMAN [6, Theorem 2] and SION and ZELMER [9, Theorem 2.2] independently proved the following result.

Theorem 5. If $(X, \mathscr{T})$ is T_1 and has a σ-point finite base then it is quasi-metrizable.

However [7, Example 1] or [9, Example 3.2] shows that this condition is not necessary. More recently, FLETCHER and LINDGREN [2] have introduced the concept of a σ-Q-base. A collection $\mathscr{C}$ of open sets in $(X, \mathscr{T})$ is called a Q-collection if $\cap\{C \in \mathscr{C} : x \in C\} \in \mathscr{T}$ for each $x \in X$. A base for $\mathscr{T}$ which is the union of a countable family of Q-collections is called a σ-Q-base for $\mathscr{T}$. FLETCHER and LINDGREN [2, Theorem 3.2] use quasi-uniform arguments to obtain the following theorem.

Theorem 6. $(X, \mathscr{T})$ is non-archimedeanly quasi-metrizable if and only if it is T_1 and has a σ-Q-base.

They then ask whether this theorem gives necessary and sufficient conditions for quasi-metrizability, and state an equivalent problem in terms of compatible quasi-uniformities. In view of Theorem 4, it seems that the question of whether every quasi-metric topological space has a compatible non-archimedean quasi-metric, can be stated as a bitopological problem.

Question. If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is quasi-metrizable is it pairwise zero dimensional?

However, the situation is not that simple because the conjugates of compatible quasi-metrics need not be compatible, as the following example shows.

Example 2. Let X be the set of positive integers and define the non-negative real valued function p on XxX by

$$p(n, m) = \begin{cases} 1 \text{ if } n < m \\ 0 \text{ if } n = m \\ (1/n) \text{ if } n > m \end{cases}$$

A discussion of cases shows that p satisfies the (strong) triangle inequality, so that p is a (non-archimedean) quasi-metric on X, with conjugate q given by

$$q(n, m) = \begin{cases} (1/m) \text{ if } n < m \\ 0 \text{ if } n = m \\ 1 \text{ if } n > m. \end{cases}$$

Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be the bitopological space induced by p and q. Now $B(n, p, (1/n)) = \{n\}$ for each $n \in X$, so that $(X, \mathscr{T}_1)$ is discrete. But $(X, \mathscr{T}_2)$ is not discrete because it is not Hausdorff. For let $m, n \in X$, $\varepsilon, \delta > 0$ and $U = B(m, q, \varepsilon)$ and $V = B(n, q, \delta)$. There is an $r \in X$ such that $r > \max \{m, n, (1/\varepsilon), (1/\delta)\}$.

Then $q(m, r) = (1/r) < \varepsilon$ and $q(n, r) = (1/r) < \delta$, so that $r \in U \cap V$. Hence there is no pair of disjoint $\mathscr{T}_2$ open sets one containing m and the other containing n. If we regard the discrete metric d on X as a quasi-metric, then being symmetric it is self conjugate, so that it induces the bitopological space $(X, \mathscr{T}_1, \mathscr{T}_1)$. Then d and p are compatible quasi-metrics on X with non-compatible conjugates. We note that both $(X, \mathscr{T}_1, \mathscr{T}_1)$ and $(X, \mathscr{T}_1, \mathscr{T}_2)$ are pairwise zero dimensional.

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